




Markov Chains on Graded Posets

Compatibility of Up-Directed and Down-Directed Transition Probabilities

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Received: 3 December 2015 / Accepted: 28 November 2016 / Published online: 8 April 2017
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Abstract We consider two types of discrete-time Markov chains where the state space is a graded poset and the transitions are taken along the covering relations in the poset. The first type of Markov chain goes only in one direction, either up or down in the poset (an *up chain* or *down chain*). The second type toggles between two adjacent rank levels (an *up-and-down chain*). We introduce two compatibility concepts between the up-directed transition probabilities (an *up rule*) and the down-directed (a *down rule*), and we relate these to compatibility between up-and-down chains. This framework is used to prove a conjecture about a limit shape for a process on Young's lattice. Finally, we settle the questions whether the reverse of an up chain is a down chain for some down rule and whether there exists an up or down chain at all if the rank function is not bounded.

Keywords Graded poset · Markov chain · Young diagram · Young's lattice · Limit shape

Supported by the Swedish Research Council, grants 2010-5565 and 621-2009-6090

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1 Introduction

A Hasse walk is a walk along the covering relations in a poset [9, 10]. In [5], Eriksson and Sjöstrand discussed how several famous models of stochastic processes can be regarded as random Hasse walks on Young's lattice, either walks that go steadily upwards (e.g. Simon's model of urban growth) or that alternately go up and down (e.g. the Moran model in population genetics). The aim of the present paper is to develop a general framework of such undirected and alternatingly directed random Hasse walks on graded posets.

Let $I \subseteq \mathbb{Z}$ be a (possibly infinite) interval of the integers. An I -graded poset Ω is a countable (or finite) poset together with a surjective map $\rho: \Omega \rightarrow I$, called the *rank function*, such that

- $u < v$ implies $\rho(u) < \rho(v)$, and
- $u \lessdot v$ implies $\rho(v) = \rho(u) + 1$,

where $u \lessdot v$ means that v covers u , that is $u < v$ but there is no w with $u < w < v$. We can partition $\Omega = \bigcup_{i \in I} \Omega_i$ into its *level sets* $\Omega_i = \rho^{-1}(i)$.

We will now describe two types of stochastic processes on Ω .

1.1 Up Chains and Down Chains

An assignment of a probability $T(u \rightarrow v)$ to any pair $(u, v) \in \Omega \times \Omega$ is an *up rule* if

- $T(u \rightarrow v) > 0 \Leftrightarrow u \lessdot v$, and
- $\sum_{v \in \Omega} T(u \rightarrow v) = 1$ for any u with non-maximal rank.

Analogously, it is a *down rule* if

- $T(u \rightarrow v) > 0 \Leftrightarrow u \gtrdot v$, and
- $\sum_{v \in \Omega} T(u \rightarrow v) = 1$ for any u with non-minimal rank.

Throughout, we shall use U and D to denote an up rule and a down rule, respectively.

Example 1 Figure 1 shows an example of a $[0, 2]$ -graded poset with two sets of probabilities forming an up rule (left) and a down rule (right).

Up rules and down rules define classes of Markov chains on Ω . We will consider such Markov chains $(X_i)_{i \in J}$ on Ω for any (possibly infinite) subinterval $J \subseteq I$.

Definition 1 Let U be an up rule and let D be a down rule on an I -graded poset Ω . A Markov chain $(X_i \in \Omega_i)_{i \in J}$ is a *U -chain (with time interval J)* if

$$\text{Prob}(X_i = u \text{ and } X_{i+1} = v) = \text{Prob}(X_i = u)U(u \rightarrow v) \quad (1)$$

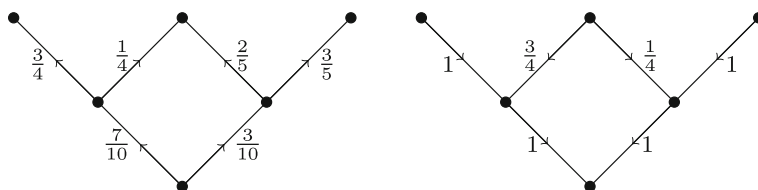


Fig. 1 An up rule (left) and a down rule (right) on some Ω

for any non-maximal $i \in J$ and any $u, v \in \Omega$. Analogously, it is a D -chain if

$$\text{Prob}(X_i = u \text{ and } X_{i-1} = v) = \text{Prob}(X_i = u)D(u \rightarrow v) \quad (2)$$

for any non-minimal $i \in J$ and any $u, v \in \Omega$.

A U - or D -chain is *maximal* if its time interval is I , and it is *positive* if $\text{Prob}(X_i = u) > 0$ for any $i \in J$ and any $u \in \Omega_i$.

In this paper we shall examine when an up rule and a down rule are compatible with each other. We shall distinguish between a weaker and a stronger notion of compatibility.

Definition 2 An up rule U and a down rule D on Ω are *compatible* if there is a maximal U -chain $(X_i \in \Omega_i)_{i \in I}$ and a maximal D -chain $(Y_i \in \Omega_i)_{i \in I}$ such that, for any $i \in I$, the random variables X_i and Y_i are equally distributed.

U and D are *strongly compatible* if there is a Markov chain that is both a maximal U -chain and a maximal D -chain.

Example 2 Let U and D be the up and down rules depicted in Fig. 2. Clearly, U and D are compatible — just assign the probability $1/2$ to each element in the poset. But they are not strongly compatible since the probability of going diagonally is $3/4$ for the up rule and $1/4$ for the down rule.

In order to study probability distributions on the poset Ω we introduce some notation.

Let $\ell_1(\Omega)$ denote the Banach space of real-valued functions f on Ω such that the norm $\|f\| = \sum_{u \in \Omega} |f(u)|$ is finite. Let

$$\ell_1(\Omega_i) = \{f \in \ell_1(\Omega) : f^{-1}(\mathbb{R} \setminus \{0\}) \subseteq \Omega_i\}$$

denote the subspace consisting of functions with support on the level set Ω_i and let

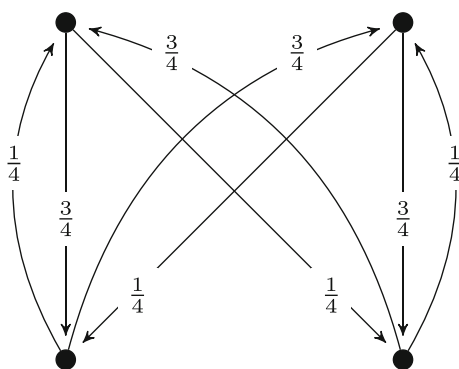
$$M(\Omega_i) = \{\pi \in \ell_1(\Omega_i) : \pi(u) \geq 0 \text{ for any } u \in \Omega_i \text{ and } \sum_{u \in \Omega_i} \pi(u) = 1\}$$

denote the set of probability distributions on Ω_i .

An up or down rule T induces a linear operator (which, by abuse of notation, also is called T) on $\ell_1(\Omega)$ defined by

$$(T\pi)(v) = \sum_{u \in \Omega} T(u \rightarrow v)\pi(u).$$

Fig. 2 The up rule (*bent arrows*) and the down rule (*straight arrows*) are compatible but not strongly compatible



Stepwise application of this operator defines sequences of probability distributions with support on one level set at a time, as follows.

Definition 3 Let U be an up rule and let D be a down rule on an I -graded poset Ω . A sequence $(\pi_i \in M(\Omega_i))_{i \in J}$ for some (possibly infinite) interval $J \subseteq I$ is a U -sequence if $U\pi_i = \pi_{i+1}$ for any non-maximal $i \in J$, and it is a D -sequence if $D\pi_i = \pi_{i-1}$ for any non-minimal $i \in J$.

A U - or D -sequence $(\pi_i)_{i \in J}$ is *positive* if $\pi_i(u) > 0$ for any $i \in J$ and any $u \in \Omega_i$.

Clearly, there is a one-to-one correspondence between U -sequences and U -chains and between D -sequences and D -chains.

Observation 1 U and D are compatible if and only if there exists a U -sequence that is also a D -sequence.

1.2 Up-and-Down Processes

Next we turn to alternatingly directed Markov chains on Ω .

Given an up rule U and a down rule D , we define a UD -chain as a Markov chain with state space Ω and with transitions induced by U and D alternately.

Definition 4 Let U be an up rule and let D be a down rule on an I -graded poset Ω . A UD -chain is a Markov chain $(X^{(0)}, X^{(1)}, \dots)$ on Ω such that

$$\text{Prob}(X^{(t)} = u \text{ and } X^{(t+1)} = v) = \begin{cases} \text{Prob}(X^{(t)} = u)U(u \rightarrow v) & \text{if } t \text{ is even} \\ \text{Prob}(X^{(t)} = u)D(u \rightarrow v) & \text{if } t \text{ is odd} \end{cases} \quad (3)$$

for any $u, v \in \Omega$.

Figure 3 depicts the level sets $\Omega_1 = \{u_1, u_2\}$ and $\Omega_2 = \{v_1, v_2, v_3\}$ of the poset Ω in Fig. 1 with the same up rule U and down rule D . The up rule U and the down rule D define a UD -chain with state space $\Omega_1 \cup \Omega_2$.

1.3 Results

Our first result relates the UD -process to the compatibility of U - and D -chains. We state this as two theorems, one about strong compatibility and one about compatibility. These theorems follow almost immediately from the definitions, but nevertheless they can be powerful

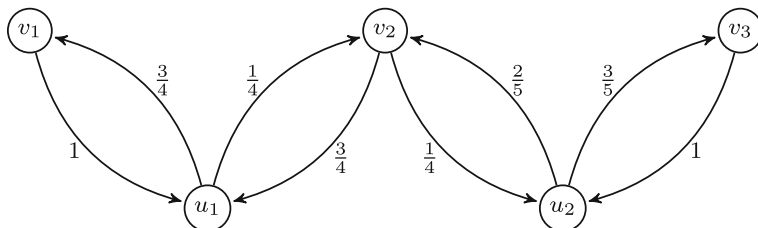


Fig. 3 A UD -chain

tools. In Section 2 we use Theorem 2 to prove a conjecture of Eriksson and Sjöstrand [5] about the limit shape of a Markov process on Young's lattice.

Theorem 1 *Let Ω be an I -graded poset with an up rule U and a down rule D , and let $(X_i \in \Omega_i)_{i \in I}$ be a Markov chain. The following are equivalent.*

- (a) $(X_i)_{i \in I}$ is both a U -chain and a D -chain (and hence U and D are strongly compatible).
- (b) For any adjacent levels $i, i+1 \in I$ it holds that $X_i, X_{i+1}, X_i, X_{i+1}, \dots$ is a UD -chain.

Proof In the sequence $(X_i, X_{i+1}, X_i, X_{i+1}, \dots)$ the even-indexed and odd-indexed variables are X_i and X_{i+1} , respectively. The two cases in Eq. 3 in Definition 4 with $t = i$ for even t and $t = i+1$ for odd t are equivalent to Eq. 1 together with Eq. 2 in Definition 1. \square

In other words, Theorem 1 says that if and only if random variables can be defined on each level set such that they correspond both to a Markov chain following the up rule and a Markov chain following the down rule, they also correspond to all Markov chains following the up and down rule alternatingly.

Theorem 2 *Let Ω be an I -graded poset with an up rule U and a down rule D , and let $(\pi_i \in M(\Omega_i))_{i \in I}$ be a sequence of probability distributions. The following are equivalent.*

- (a) $(\pi_i)_{i \in I}$ is both a U -sequence and a D -sequence (and hence U and D are compatible).
- (b) For any adjacent levels $i, i+1 \in I$ it holds that $(\pi_i + \pi_{i+1})/2$ is a stationary distribution of the UD -process.

Proof Note that $(\pi_i + \pi_{i+1})/2$ is a stationary distribution of the UD -process if and only if $U\pi_i = \pi_{i+1}$ and $D\pi_{i+1} = \pi_i$. Now the theorem follows directly from Definition 3 (with $J = I$). \square

In words, Theorem 2 says that a given sequence of probability distributions is both a U -sequence and a D -sequence if and only if the average (in $\ell_1(\Omega)$) of two adjacent such distributions is a stationary distribution of the UD -process.

The second part of this paper will answer the following questions that arise naturally from the notion of compatibility.

- Q1. Given an up rule U and a U -chain (X_i) , is there a down-rule D such that (X_i) is a D -chain?
- Q2. Given an up or down rule T , is there a maximal T -chain?

The answer to Q1 is that every *positive* U -chain is a D -chain for some D . This is Theorem 3 in Section 3 and we give a constructive proof using Bayesian updating of probability distributions between adjacent levels.

The answer to Q2 is more complex and is presented as three theorems in Section 4. If all level sets Ω_i are finite, it turns out that there always exists a T -chain, but not necessarily a positive one. The proofs rely on topological arguments and require the axiom of choice.

1.4 Applications

We will apply the results to processes on two particular posets: (i) Young's lattice; and (ii) the d -dimensional nonnegative integer lattice.

In [5], Eriksson and Sjöstrand study stochastic processes on Young diagrams of integer partitions, in particular their limit shapes. In the framework of this paper, these processes are up processes and up-and-down processes, respectively, where the underlying poset is Young's lattice. A motivation for this work is a limit shape conjecture in [5] for a certain up-and-down process (described in Section 1.5 below). Using the results in the current paper, in particular Theorem 2, we will prove this conjecture. This is done in Corollary 2 in Section 2.

Applications of our theory to the d -dimensional nonnegative integer lattice are presented in Section 3.1.

1.5 An Application of Theorem 2 to a Process on Young's Lattice

One of the processes studied in [5], called DEROW-ROW(μ), is a UD -chain on Young's lattice. We will first introduce some notation for integer partitions and Young diagrams and then describe the up rule ROW(μ) and the down rule DEROW used in this process.

1.5.1 Notation

An introduction to the theory of integer partitions can be found in [1].

With \mathcal{P}_n we mean the set of all partitions of the positive integer n . For $\lambda \in \mathcal{P}_n$, we write $|\lambda| = n$. Denote the parts of the partition by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ and $N = N(\lambda)$ is the number of parts of λ . Let $r_i = r_i(\lambda) \geq 0$ denote the number of parts of size i . Thus, $N = \sum_{i=1}^n r_i$ and $n = \sum_{i=1}^N \lambda_i = \sum_{i=1}^n i r_i$.

An integer partition λ can be represented by a Young diagram drawn as left-aligned rows of squares in the first quadrant such that the i th row from the bottom has length λ_i . The diagrams in the first three levels of Young's lattice can be seen in Fig. 4. With $\lambda \in \mathcal{P}_n$ we mean either the partition or its corresponding Young diagram.

1.5.2 Description of the Process

To describe the UD -chain DEROW-ROW(μ), we need to associate squares with row-lengths and row-lengths with corners in a Young diagram as follows.

Definition 5 See Fig. 5. Consider some given Young diagram λ . For any of its squares s let $\kappa(s)$ denote the length of the row to which s belongs. If κ is a row length, let $\omega(\kappa)$ and $\iota(\kappa)$

Fig. 4 Young's lattice

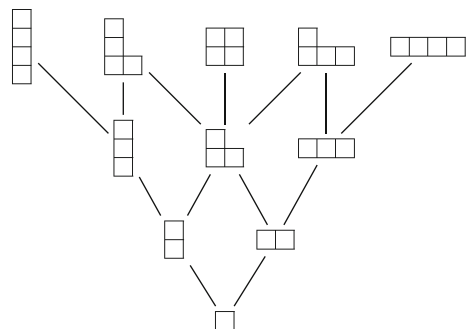
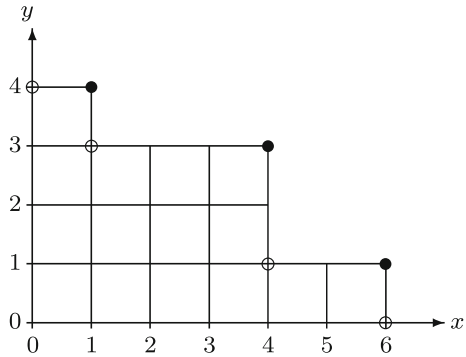


Fig. 5 Inner and outer corners in the Young diagram of the partition $\lambda = (6, 4, 4, 1)$. The outer corners (filled circles) are from left to right $\omega(1)$, $\omega(4)$ and $\omega(6)$. The inner corners (unfilled circles) are from left to right $\iota(0)$, $\iota(1)$, $\iota(4)$ and $\iota(6)$



denote the unique outer corner and inner corner, respectively, for which the row coordinate is κ :

$$\begin{aligned}\omega(\kappa) &= (\kappa, \max\{i \mid \lambda_i = \kappa\}), \\ \iota(\kappa) &= \begin{cases} (\kappa, \max\{i \mid \lambda_i > \kappa\}) & \text{if } \kappa < \lambda_1 \\ (\lambda_1, 0) & \text{if } \kappa = \lambda_1 \end{cases}.\end{aligned}$$

Consider a current Young diagram λ . The action of the down rule DEROW is defined by choosing a non-empty row i uniformly at random and removing the corresponding outer corner $\omega(\lambda_i)$.

The action of the up rule ROW(μ) is defined as follows: With probability μ create a new row of length 1. Otherwise make a uniformly random choice of a row i among the $N(\lambda)$ non-empty rows and insert a new square at the corresponding inner corner $\iota(\lambda_i)$. See Definition 5.

The process DEROW-ROW(μ) is the up-and-down process on \mathcal{P}_n (for some $n \geq 2$) using these up and down rules.

1.5.3 The Stationary Distributions

Lemma 1 For any $0 < \mu < 1$, the stationary distribution over the partitions in $\mathcal{P}_i \cup \mathcal{P}_{i+1}$ in the process ROW-DEROW(μ) is

$$\pi_i^{UD}(\lambda) = \frac{1}{2}(1 - \mu)^{|\lambda| - N(\lambda)} \mu^{N(\lambda) - 1} \frac{N(\lambda)!}{\prod_k r_k(\lambda)!}. \quad (4)$$

Proof In [5] (equations (12) and (13)) it is proved that, up to a normalization constant, $\pi_i^{UD}(\lambda)$ is given by

$$\pi_i^{UD}(\lambda) = \frac{\left(\frac{\mu}{1-\mu}\right)^{N(\lambda)} N(\lambda)!}{\prod_k r_k(\lambda)!} \cdot \begin{cases} \frac{1}{2}c_i & \text{if } |\lambda| = i + 1, \\ \frac{1}{2}c_i/(1 - \mu) & \text{if } |\lambda| = i. \end{cases} \quad (5)$$

for some constant c_i (independent of λ). It follows that $c_{i+1}/(1 - \mu) = c_i$ for each i , and since $c_1 = (1 - \mu)^2/\mu$ we must have $c_i = (1 - \mu)^{i+1}/\mu$. Inserting this expression for c_i in Eq. 5 yields Eq. 4. \square

Combining the lemma with Theorem 2 yields a formula for the distributions of the U -chain induced by the up rule $\text{ROW}(\mu)$.

Corollary 1 *The up rule $\text{ROW}(\mu)$ and the down rule DEROW are compatible and the distribution of the $\text{ROW}(\mu)$ process on \mathcal{P}_i is given by the probability function*

$$p(\lambda) = (1 - \mu)^{i - N(\lambda)} \mu^{N(\lambda) - 1} \frac{N(\lambda)!}{\prod_k r_k(\lambda)!}.$$

In fact the derivation of Eqs. 12 and 13 in [5] reveals that the stationary $\text{DEROW-ROW}(\mu)$ process is reversible, and hence, by Theorem 1, $\text{ROW}(\mu)$ and DEROW are strongly compatible.

2 Limiting Objects

The original motivation for the current work has been the study of limiting objects, in particular limit shapes of Young diagrams under stochastic processes as initiated in [5]. As a corollary to Theorem 2, we can now prove a limit shape conjecture in [5] (Conjecture 2).

First, let us remind ourselves what is meant by a *scaling* of a Young diagram. As the number of squares n grows we need to rescale the diagram to achieve any limiting behaviour. Following [5] and [11], a diagram is rescaled using a *scaling factor* $a_n > 0$ such that all row lengths are multiplied by $1/a_n$ and all column heights are multiplied by a_n/n , yielding a constant diagram area of 1.

Corollary 2 *Let $\mu_n \log(\mu_n n) \rightarrow 0$ and $\mu_n n \rightarrow \infty$ as $n \rightarrow \infty$ and choose the scaling $a_n = 1/\mu_n$. Then the stationary distribution of the $\text{DEROW-ROW}(\mu_n)$ process has the limit shape*

$$y(x) = e^{-x}.$$

Proof By Corollary 1, the stationary distribution for $\text{DEROW-ROW}(\mu)$ equals the distribution for $\text{ROW}(\mu)$. By Theorem 4 in [5], the limit shape for this distribution with $\mu = \mu_n$ is $y = e^{-x}$ as long as $\mu_n \log(\mu_n n) \rightarrow 0$ and $\mu_n n \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, under these conditions the limit shape for $\text{DEROW-ROW}(\mu_n)$ must also be $y = e^{-x}$. \square

2.1 The Asymptotics of μ_n in $\text{DEROW-ROW}(\mu_n)$

Theorem 4 in [5] has the conditions $\mu_n n \rightarrow \infty$ and $\mu_n \log(\mu_n n) \rightarrow 0$ as $n \rightarrow \infty$. After scaling a Young diagram with scaling factor $a_n = 1/\mu_n$, each square will have width $1/a_n = \mu_n$ and height $a_n/n = \frac{1}{\mu_n n}$. In order for the boundary of the Young diagram (which is the object whose limiting behaviour one considers) to resolve properly as $n \rightarrow \infty$, we must at least have

$$\mu_n \rightarrow 0 \text{ and } \mu_n n \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (6)$$

However, Theorem 4 in [5] uses the stronger assumption $\mu_n \log(\mu_n n) \rightarrow 0$ as $n \rightarrow \infty$. Whether this can be relaxed in Corollary 2 is an open problem.

2.2 A Generalization of Limiting Objects

One may also generalize the concepts of limiting object and limit distribution. In order to be able to talk about a generic process having a limiting object and a limit distribution, let S be a separable metric space, called a *limit space*, with the metric d_S .

For $n = 0, 1, \dots$ let R_n be the countable set of states reachable after n steps, and let $f_n: R_n \rightarrow S$ be a “scaling” function. A stochastic process $(X_n \in R_n)_{n=0}^\infty$ having a limit distribution (with respect to the metric d_S) corresponds to the convergence of the random variables $\{f_n(X_n)\}$ in *distribution* to a random variable $X \in S$; the distribution of X is the limit distribution.

Further, if the limit distribution has all probability mass concentrated in a single point in S , in other words, if X is constant, then this point is the *limiting object* of the process (in which case we may equivalently talk about convergence in *probability* of $\{f_n(X_n)\}$ to the limiting object).

In this paper, we have made this concept tangible by studying processes on Young’s lattice, where the limit space is the set of decreasing functions in the first quadrant with integral 1. In the next section, we will study processes on the d -dimensional nonnegative integer lattice \mathbb{N}^d where the limit space is the set of nonnegative real d -tuples adding to 1. We see a further possible application to this generalization in the study of limits of permutation sequences (see for example [7]). In this case S is the set of Lebesgue measurable functions $[0, 1]^2 \rightarrow [0, 1]$ with certain properties.

3 Given an Up Rule, is there a Compatible Down Rule?

In this section we will prove the existence of a down rule compatible with a given up rule. The proof is constructive and we will use the construction in some examples.

Theorem 3 *Let U be an up rule on an I -graded poset Ω , and let $(X_i)_{i \in I}$ be a U -chain. Then (X_i) is a D -chain for some down rule D on Ω if and only if $\text{Prob}(X_i = u) = 0$ implies $\text{Prob}(X_{i+1} = v) = 0$ for any $i, i + 1 \in I$ and any $\Omega_i \ni u \leq v \in \Omega_{i+1}$. In particular this holds when (X_i) is positive.*

Proof By Theorem 1, (X_i) is a D -chain if and only if $X_i, X_{i+1}, X_i, X_{i+1}, \dots$ is a UD -chain for any $i \in I$. Since $X_i, X_{i+1}, X_i, X_{i+1}, \dots$ is a reversible Markov chain (see any text book on Markov chains, for instance [8]), it is a UD -chain if and only if $\text{Prob}(X_i = u)U(u \rightarrow v) = \text{Prob}(X_{i+1} = v)D(v \rightarrow u)$ for any $u, v \in \Omega$. A D satisfying that equation can be chosen by letting

$$D(v \rightarrow u) = \frac{\text{Prob}(X_i = u)U(u \rightarrow v)}{\text{Prob}(X_{i+1} = v)} \quad (7)$$

but this is possible only unless $\text{Prob}(X_i = u) = 0$ and $\text{Prob}(X_{i+1} = v) > 0$ for some $u \leq v$ or $\text{Prob}(X_i = u) > 0$ and $\text{Prob}(X_{i+1} = v) = 0$ for some $u \leq v$. The latter is impossible since $\text{Prob}(X_{i+1} = v) \geq \text{Prob}(X_i = u)U(u \rightarrow v)$. \square

3.1 Processes on the d -Dimensional Nonnegative Integer Lattice

In this section we will demonstrate two applications of Theorem 3 on the d -dimensional nonnegative integer lattice \mathbb{N}^d . We will construct down rules compatible with given up rules. We shall also see examples of processes both with and without a limiting object.

3.1.1 The Lattice \mathbb{N}^d

For a positive integer d , let (\mathbb{N}^d, \leq) be the poset of nonnegative integer d -tuples (x_1, \dots, x_d) ordered component-wise, i.e. for $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{N}^d$, we have $x \leq y$ if $x_i \leq y_i$ for all $i = 1, \dots, d$. For $x, y \in \mathbb{N}^d$, we have

$$\begin{aligned} x \vee y &= (\max(x_1, y_1), \dots, \max(x_d, y_d)) \text{ and} \\ x \wedge y &= (\min(x_1, y_1), \dots, \min(x_d, y_d)), \end{aligned}$$

so \mathbb{N}^d is a lattice, which we will refer to as *the d -dimensional nonnegative integer lattice*.

Obviously, \mathbb{N}^d is graded with the rank function

$$|x| = \sum_{j=1}^d x_j$$

and for $n \geq 0$, the level set $\mathbb{N}_n^d = \{x \in \mathbb{N}^d : |x| = n\}$ is the set of weak compositions of n .

3.1.2 A Limiting Object on \mathbb{N}^d

As an analogy to the concepts of limit shapes for birth- and birth-and-death processes on Young diagrams we will here define a limiting object for processes on \mathbb{N}^d .

If we divide the coordinates of a lattice point reached in an up process on \mathbb{N}^d by the number of steps taken in the process, the result is a point in the $(d-1)$ -dimensional simplex $\Delta^{d-1} = \{(x_1, \dots, x_d) \in [0, 1]^d \mid x_1 + \dots + x_d = 1\}$. Under this scaling we define a limiting object as follows.

Definition 6 For an up process on \mathbb{N}^d , let $X_n = (X_n^1, \dots, X_n^d)$ be the lattice point after n steps. A point $x \in \Delta^{d-1}$ is a limit point of the up process if for any $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\left| \frac{1}{n} X_n - x \right| < \varepsilon \right) = 1.$$

3.1.3 Notation

For $x = (x_1, \dots, x_d) \in \mathbb{N}^d$, let

$$\begin{aligned} x_{(i)} &= (x_1, \dots, x_i - 1, \dots, x_d) \text{ and} \\ x^{(i)} &= (x_1, \dots, x_i + 1, \dots, x_d) \end{aligned}$$

for $i = 1, \dots, d$. An up process on \mathbb{N}^d starts at $(0, \dots, 0)$. In each step, the up rule U increases the rank of the current state by 1, i.e. increments exactly one component. Thus, U is determined by the probabilities $U(x \rightarrow x^{(i)})$ for $i = 1, \dots, d$.

For $n \in \mathbb{N}$, let $p_n: \mathbb{N}_n^d \rightarrow [0, 1]$ be the probability function on \mathbb{N}_n^d induced by the up rule U .

3.1.4 An Up Process on \mathbb{N}^d with a Limiting Object

Consider the up rule U on \mathbb{N}^d governed by the constant transition probabilities $U(x \rightarrow x^{(i)}) = v_i$. We may use Theorem 3 to conclude that there is a down rule D compatible with U . Let us construct it!

First of all, the probability to reach $x = (x_1, \dots, x_d)$ (after $n = |x|$ steps in the up process induced by U) is clearly

$$p_n(x_1, \dots, x_d) = \frac{n!}{x_1! \cdots x_d!} v_1^{x_1} \cdots v_d^{x_d}.$$

As a consequence,

$$p_{n-1}(x_{(i)}) = \frac{(n-1)!}{x_1! \cdots (x_i-1)! \cdots x_d!} v_1^{x_1} \cdots v_i^{x_i-1} \cdots v_d^{x_d}.$$

We use Eq. 7 to compute the transition probability $D(x \rightarrow x_{(i)})$ from $x \in \mathbb{N}_n^d$ to $x_{(i)} \in \mathbb{N}_{n-1}^d$:

$$D(x \rightarrow x_{(i)}) = \frac{p_{n-1}(x_{(i)})U(x_{(i)} \rightarrow x)}{p_n(x)} = \frac{p_{n-1}(x_{(i)})v_i}{p_n(x)} = \frac{x_i}{n} = \frac{x_i}{|x|}.$$

Here we see that an up rule employing the degree of freedom parameters v_1, \dots, v_d has a compatible down rule with no such degree of freedom present. As we saw in Corollary 1, this is also the case with the up rule in the process $\text{ROW}(\mu)$ having a compatible down rule DEROW , void of the degree of freedom parameter μ .

Proposition 1 *The up process on \mathbb{N}^d using the up rule $U(x \rightarrow x^{(i)}) = v_i$ has the limit point (v_1, \dots, v_d) .*

Proof For this process we have $\text{Prob}(X_n^i = k) = \binom{n}{k} v^k (1-v)^{n-k}$ for $i = 1, \dots, d$, i.e. $X_n^i \sim \text{Bin}(n, v_i)$, which means $E(X_n^i)/n = v_i$. Thus, the limit point is the d -tuple (v_1, \dots, v_d) . \square

3.1.5 An Up Process on \mathbb{N}^d Without a Limiting Object

We will now consider an up process on \mathbb{N}^d starting at $(0, \dots, 0)$ induced by an up rule that depends on the current level.

Proposition 2 *The process on \mathbb{N}^d induced by the up rule U governed by the transition probabilities*

$$U(x \rightarrow x^{(i)}) = \frac{x_i + 1}{x_1 + \dots + x_d + d} = \frac{x_i + 1}{|x| + d} \quad (8)$$

has a uniform distribution on \mathbb{N}_n^d , for all $n \geq 0$.

Proof We prove this by induction over the number of steps n in the process. First of all we observe that $|\mathbb{N}_n^d| = \binom{d+n-1}{d-1}$.

For $n = 0$, we have a trivial uniform distribution.

For $n \geq 0$, assume that the distribution on \mathbb{N}_n^d under this process is uniform, i.e., $p_n(x) = |\mathbb{N}_n^d|^{-1} = \binom{d+n-1}{d-1}^{-1}$ for all $x \in \mathbb{N}_n^d$. We want to prove that the distribution over \mathbb{N}_{n+1}^d is also uniform, i.e., $p_{n+1}(x) = |\mathbb{N}_{n+1}^d|^{-1} = \binom{d+n}{d-1}^{-1}$.

Let $x = (x_1, \dots, x_d) \in \mathbb{N}_{n+1}^d$ be a fixed element. Of course $|x| = n + 1$. Now, x can be reached from any of the elements $x_{(1)}, \dots, x_{(d)} \in \mathbb{N}_n^d$. By Eq. 8,

$$U(x_{(j)} \rightarrow x) = \frac{(x_j - 1) + 1}{|x_{(j)}| + d} = \frac{x_j}{n + d},$$

and by the induction hypothesis, $p_n(x_{(j)}) = \binom{d+n-1}{d-1}^{-1}$, so the probability $p_{n+1}(x)$ for $x \in \mathbb{N}_{n+1}^d$ is

$$\begin{aligned} p_{n+1}(x) &= \sum_{j=1}^d p_n(x_{(j)}) P(x_{(j)} \rightarrow x) \\ &= \binom{d+n-1}{d-1}^{-1} \sum_{j=1}^d \frac{x_j}{n+d} \\ &= \frac{(d-1)!n!(n+1)}{(d+n-1)!(n+d)} = \frac{(d-1)!(n+1)!}{(n+d)!} = \binom{d+n}{d-1}^{-1}, \end{aligned}$$

so the distribution is uniform also on \mathbb{N}_{n+1}^d , and the result follows by induction. \square

Since the distribution is uniform, this process cannot have a limiting object.

As in Section 3.1.4, let us use Eq. 7 to construct the down rule compatible with the up rule in Theorem 2. We get the probability $D(x \rightarrow x_{(i)})$ for moving from $x \in \mathbb{N}_n^d$ to $x_{(i)} \in \mathbb{N}_{n-1}^d$ by

$$\begin{aligned} D(x \rightarrow x_{(i)}) &= \frac{p_{n-1}(x_{(i)}) P(x_{(i)} \rightarrow x)}{p_n(x)} \\ &= \frac{\binom{d+n-2}{d-1}^{-1} \frac{x_i}{n-1+d}}{\binom{d+n-1}{d-1}^{-1}} = \frac{\binom{d+n-1}{d-1}}{\binom{d+n-2}{d-1}} \frac{x_i}{n-1+d} = \frac{x_i}{n} = \frac{x_i}{|x|}. \end{aligned}$$

4 Given an Up or Down Rule T , is there a Maximal T -Chain?

Let T be an up or down rule on an I -graded poset Ω . Recall that there is a one-to-one correspondence between T -chains and T -sequences, so questions about the existence of maximal T -chains are equivalent to questions about the existence of maximal T -sequences.

If I has a minimal element m there obviously exists a maximal U -sequence for a given up rule U : Just choose any probability distribution $\pi_m \in M(\Omega_m)$ on level m , and the up rule U will induce a U -sequence $(U^{i-m}\pi_m)_{i \in I}$. However, if I has no lower bound it is not obvious whether there exists a maximal U -sequence, and, by symmetry, if I has no upper bound it is not obvious whether there exists a maximal D -sequence for a given down rule D . Here is an example where no such D -sequence exists.

Example 3 Let Ω be the two-dimensional integer lattice \mathbb{Z}^2 with the partial order $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. It is \mathbb{Z} -graded by $\rho(x, y) = x + y$. Let D be the down rule with probability $1/2$ for each edge in the Hasse diagram. If we start at any element of high rank n and follow down edges randomly according to the down rule, the chance of hitting any particular element of rank zero is very small; it tends to zero as n grows. Thus, in a maximal D -sequence any rank-zero element must be given the probability zero, which is impossible.

The phenomenon in the above example cannot happen if the level sets Ω_i are finite, and, as we will see in Theorem 6, in this case there is always a maximal T -sequence. As the following example reveals, however, the existence of a *positive* maximal T -sequence is not guaranteed.

Example 4 Look at the two-dimensional nonnegative integer lattice \mathbb{N}^2 with the down rule given by the probability $1/2$ at every edge in the Hasse diagram, except for the leftmost and rightmost edges which must have probability one; see Fig. 6. If we start at any element of high rank n and follow down edges randomly according to the down rule, with very high probability we will walk into the left or right border before we reach level 2. Thus, the probability of reaching the middle element $(1, 1)$ at level 2 tends to zero as n grows. This means that every D -chain (X_i) must have $\text{Prob}(X_2 = (1, 1)) = 0$.

We will present three theorems about the existence of T -sequences and positive T -sequences. Their proofs all depend on the following lemma.

Lemma 2 Let T be an up or down rule on an I -graded poset Ω . Let $(C_i \subseteq M(\Omega_i))_{i \in I}$ be a sequence of compact sets and suppose for any $m \leq n$ in I there exists a T -sequence $(\pi'_i \in C_i)_{i \in [m, n]}$. Then there exists a maximal T -sequence $(\pi_i \in C_i)_{i \in I}$.

Proof Without loss of generality, we may assume that $T = U$ is an up rule.

Let Π be the product space $\Pi = \prod_{i \in I} C_i$. For any non-maximal $k \in I$, let

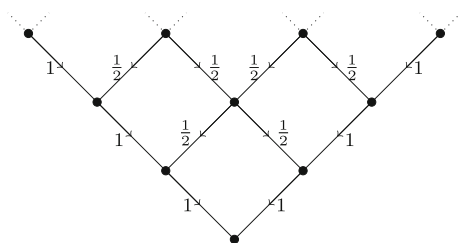
$$\Pi_k := \{(\pi_i) \in \Pi \mid U\pi_k = \pi_{k+1}\}.$$

Let us first show that the set Π_k is closed.

Clearly, Π_k is homeomorphic to the product of $\prod_{i \in I \setminus \{k, k+1\}} C_i$ and the graph $G \subseteq C_k \times C_{k+1}$ of the restriction of U to $C_k \cap U^{-1}C_{k+1}$. The map U has operator norm 1, so it is continuous. Hence, the preimage $U^{-1}C_{k+1}$ is closed and so is $C_k \cap U^{-1}C_{k+1}$. It follows that the graph G is closed and hence Π_k is closed, being homeomorphic to a product of closed sets.

By Tychonoff's theorem Π is compact, and therefore the closed subsets Π_k are compact, and so is the intersection Λ of all Π_k . Clearly, Λ is precisely the set of maximal U -sequences, so our task is to show that Λ is nonempty. To that end, suppose it is empty and define $U_k = \Pi \setminus \Pi_k$. Then $\bigcup U_k = \Pi \setminus \bigcap \Pi_k = \Pi$ so the family of all sets U_k is an open cover of Π . Since Π is compact there is an open subcover $\{U_k\}_{k \in F}$, where F is a finite set of non-maximal elements in I . Choose m and n in I so that $m \leq k < n$ for any $k \in F$. Now, by the assumption in the theorem there is a U -sequence $(\pi'_i \in C_i)_{i \in [m, n]}$. For $i \in I \setminus [m, n]$, let π'_i be an arbitrary element in C_i . The sequence $(\pi'_i)_{i \in I}$ so obtained is a

Fig. 6 The down rule on \mathbb{N}^2 in Example 4



point outside the union $\bigcup_{k \in F} U_k$, which contradicts the fact that $\{U_k\}_{k \in F}$ covers Π . Hence, our supposition that Λ is empty is false. \square

Our first existence theorem states that if there exists a T -sequence for any finite subinterval of I , and these sequences are uniformly bounded in a certain sense, then there exists a maximal T -sequence.

For $f, g \in \ell_1(\Omega)$ we will write $f \leq g$ if $f(u) \leq g(u)$ for all $u \in \Omega$.

Theorem 4 *Let T be an up or down rule on an I -graded poset Ω .*

Suppose there is a sequence $(\hat{b}_i \in \ell_1(\Omega_i))_{i \in I}$ such that for any $m \leq n$ there exists a T -sequence $(\pi_i)_{i \in [m, n]}$ with $\pi_i \leq \hat{b}_i$ for any $i \in [m, n]$. Then there exists a maximal T -sequence.

Proof For any $i \in I$, let $C_i = \{\pi \in M(\Omega_i) : \pi \leq \hat{b}_i\}$. If we can show that C_i is compact, the theorem will follow from Lemma 2. We can write $C_i = M(\Omega_i) \cap L_i$ where

$$L_i = \{\pi : \Omega_i \rightarrow \mathbb{R} \mid 0 \leq \pi \leq \hat{b}_i\} \subset \ell_1(\Omega_i).$$

Since $M(\Omega_i)$ is closed it suffices to show that L_i is compact. By the dominated convergence theorem L_i is homeomorphic to the product space $\prod_{u \in \Omega_i} [0, \hat{b}_i(u)]$ which is compact by Tychonoff's theorem. \square

For the existence of a *positive* maximal T -sequence, it is not enough to assume the existence of finite T -sequences that are uniformly bounded from above; they must be uniformly bounded from above and from below simultaneously!

Let $\ell_1^+(\Omega_i) = \{f \in \ell_1(\Omega_i) : f(u) > 0 \text{ for any } u \in \Omega_i\}$ denote the set of strictly positive ℓ_1 -functions on Ω_i .

Theorem 5 *Let T be an up or down rule on an I -graded poset Ω .*

Suppose there are sequences $(\check{b}_i \in \ell_1^+(\Omega_i))_{i \in I}$ and $(\hat{b}_i \in \ell_1^+(\Omega_i))_{i \in I}$ such that for any $m \leq n$ in I there exists a T -sequence $(\pi_i)_{i \in [m, n]}$ with $\check{b}_i \leq \pi_i \leq \hat{b}_i$ for any $i \in [m, n]$. Then there exists a positive maximal T -sequence.

Proof The proof is completely analogous to that of Theorem 4, but with $L_i = \{\pi : \Omega_i \rightarrow \mathbb{R} : \check{b}_i \leq \pi \leq \hat{b}_i\} \subset \ell_1(\Omega_i)$. \square

In most combinatorial applications, the level sets Ω_i are finite. In that case, the uniform upper bound (\hat{b}_i) in the assumption in Theorem 4 automatically exists. Furthermore, the requirement of uniformity of the lower bound (\check{b}_i) in Theorem 5 can be relaxed.

The following theorem is stated for an up rule, but the dual statement for a down rule is of course equivalent.

Theorem 6 *Let U be an up rule on an I -graded poset Ω with finite level sets Ω_i . Then there exists a maximal U -sequence, and there exists a positive maximal U -sequence if and only if there is a sequence $(b_i : \Omega_i \rightarrow (0, \infty))_{i \in I}$ with the property that for any $m \leq n$ in I there is a distribution $\pi_{m, n} \in M(\Omega_m)$ such that $U^{n-m} \pi_{m, n} \geq b_n$.*

Proof Define a sequence $(\hat{b}_i \in \ell_1(\Omega_i))_{i \in I}$ by letting $\hat{b}_i(u) = 1$ for any $u \in \Omega_i$. Now Theorem 4 guarantees the existence of a maximal U -sequence.

Next, suppose there is a sequence $(b_i : \Omega_i \rightarrow (0, \infty))_{i \in I}$ and distributions $(\pi_{m,n} \in M(\Omega_m))_{m \leq n}$ with the property given in the theorem. Let $(\gamma_i)_{i \in I}$ be positive numbers adding to one, and for each $i \in I$ put $\check{b}_i = \gamma_i b_i$.

Now, fix $m \leq n$ in I . Let $\Gamma = \sum_{i \in [m,n]} \gamma_i$ and define

$$\pi = \frac{1}{\Gamma} \sum_{i \in [m,n]} \gamma_i \pi_{m,i}.$$

Since U is a linear operator, for any $j \in [m, n]$ we have

$$U^{j-m} \pi = \frac{1}{\Gamma} \sum_{i \in [m,n]} \gamma_i U^{j-m} \pi_{m,i} \geq \frac{\gamma_j}{\Gamma} U^{j-m} \pi_{m,j} \geq \frac{\gamma_j}{\Gamma} b_j = \frac{\check{b}_j}{\Gamma} \geq \check{b}_j$$

and hence $(U^{j-m} \pi)_{j \in [m,n]}$ is a T -sequence such that $U^{j-m} \pi \geq \check{b}_j$ for any $j \in [m, n]$. Theorem 5 now yields the existence of a positive maximal U -sequence. \square

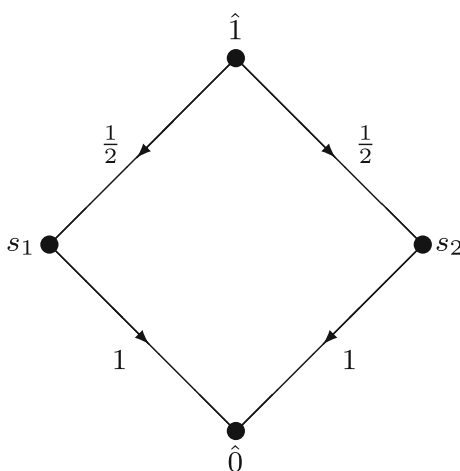
Theorem 6 guarantees the existence of a maximal T -sequence if all level sets are finite, but in general it is not possible to extend a T -sequence for a subinterval $J \subset I$ to a maximal T -sequence. For example, consider the poset in Fig. 7 with a defined down rule D . The distributions $\pi_0 \in M(\Omega_0)$ and $\pi_1 \in M(\Omega_1)$ given by $\pi_0(\hat{0}) = 1$, $\pi_1(s_1) = 1/4$ and $\pi_2(s_2) = 3/4$ constitute a D -chain, but this D -chain can obviously not be extended to the top level.

5 General Homogeneous Markov Chains

Finally, we shall use Theorem 4 to prove a more general result about the existence of Markov chains with time interval \mathbb{Z} . As far as we know, this has not been treated before.

Let S be a countable (or finite) state space and let P be a transition function on S , that is, an assignment of a probability $P(s \rightarrow s')$ to each pair $(s, s') \in S \times S$ such that $\sum_{s' \in S} P(s \rightarrow s') = 1$ for any $s \in S$. For any initial (random) state X_0 , there

Fig. 7 A graded poset with a given down rule D



is a unique Markov chain X_0, X_1, \dots with transition function P , but is there a Markov chain $\dots, X_{-1}, X_0, X_1, \dots$ with time interval \mathbb{Z} ? The following “homogeneous” version of Theorem 4 answers that question.

Theorem 7 *Let S be a state space and let P be a transition function on S . Suppose there exists a sequence of functions $(\hat{b}_i \in \ell_1(S))_{i \in \mathbb{Z}}$ such that for any integers $m \leq n$ there is a Markov chain X_m, \dots, X_n on S with transition function P such that $\text{Prob}(X_i = s) \leq \hat{b}_i(s)$ for any $i \in [m, n]$ and any $s \in S$. Then there exists a Markov chain $\dots, X_{-1}, X_0, X_1, \dots$ on S with transition function P .*

Proof Define a \mathbb{Z} -graded poset $\Omega = S \times \mathbb{Z}$ with $(s, i) \leq (s', i + 1)$ if $P(s, s') > 0$ and define an up rule U on Ω by letting

$$U((s, i) \rightarrow (s', i + 1)) = P(s \rightarrow s')$$

for any $i \in \mathbb{Z}$ and any $s, s' \in S$. Now the theorem follows from Theorem 4. \square

6 Discussion

We have applied the results in this paper to Young’s lattice (in Section 2) and to the d -dimensional nonnegative integer lattice (in Section 3.1).

As mentioned in Section 2.2, a possible application to our generalization of limiting object is in the study of sequences of permutations [7]. Similarly to the well-studied *Tsetlin library Markov chain* (see for example [6]), such a sequence can be generated by a stochastic process in which the number $n + 1$ is inserted into a permutation of $\{1, 2, \dots, n\}$ to create a permutation of $\{1, 2, \dots, n + 1\}$, wherein each of the possible $n + 1$ “slots” is assigned a probability to create a probability distribution. In terms of this paper, such probabilities constitute an up-rule U on the \mathbb{N} -graded poset Ω , where each level set $\Omega_i = \mathfrak{S}_i$ is the set of permutations of $\{1, 2, \dots, i\}$. One direction for future research is to investigate the relation between U and possible limiting objects of resulting permutation sequences in terms of [7].

Other Markov chains whose state-space has a partial order occur in the study of certain semigroups called *left-regular bands* (for example [2]) as well as in the study of poset block structures [3, 4]. Another direction for future research may thus include investigating whether the results of the present paper may be applied also to these families of Markov chains.

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References

1. Andrews, G.E., Eriksson, K.: *Integer Partitions*. Cambridge University Press (2004)
2. Brown, K.S.: Semigroups, rings, and Markov chains. *J. Theor. Probab.* **13**(3), 871–938 (2000)

3. D'Angeli, D., Donno, A.: Markov chains on orthogonal block structures. *Eur. J. Comb.* **31**(1), 34–46 (2010)
4. D'Angeli, D., Donno, A.: The lumpability property for a family of Markov chains on poset block structures. *Adv. Appl. Math.* **51**(3), 367–391 (2013)
5. Eriksson, K., Sjöstrand, J.: Limiting shapes of birth-and-death processes on Young diagrams. *Adv. Appl. Math.* **48**(4), 575–602 (2012). doi:[10.1016/j.aam.2011.12.001](https://doi.org/10.1016/j.aam.2011.12.001)
6. Fill, J.A.: An exact formula for the move-to-front rule for self-organizing lists. *J. Theor. Probab.* **9**(1), 113–160 (1996)
7. Hoppen, C., Kohayakawa, Y., Moreira, C.G., Ráth, B., Sampaio, R.M.: Limits of permutation sequences. *Journal of Combinatorial Theory, Series B* **103**(1), 93–113 (2013)
8. Ross, S.M.: *Introduction to Probability Models*. Academic Press (2014)
9. Stanley, R.P.: Differential posets. *J. Amer. Math. Soc.* **1**(4), 919–961 (1988)
10. Stanley, R.P.: *Enumerative Combinatorics*, vol. 1. Wadsworth and Brooks/Cole, Pacific Grove, CA (1996). 1986; second printing
11. Vershik, A.: Statistical mechanics of combinatorial partitions, and their limit shapes. *Funct. Anal. Appl.* **30**(2), 90–105 (1996)